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On unsteady magnetic boundary layer theory

KRISHNA LAL

G/3, *Beyond Ladies Colony, Banaras Hindu University, India*

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In the present paper, we wish to discuss the influence of a constant horizontal magnetic field upon the free convection hydromagnetic flow from a vertical infinite flat plate with variable suction velocity when the plate temperature varies with time about a non-zero constant mean. This problem has been analysed on the assumption that the fluid has a very small electric conductivity so that the perturbation in the magnetic field due to the electric current flowing in the field may be neglected. The rate of heat transfer from the plate to the fluid has been shown to lead by certain phase angle which is 45° for constant suction and decreases slowly as the unsteady part of the suction velocity increases.

1 BASIC EQUATIONS

Let x-axis be along the plate vertically upwards and the $\bar{y}$-axis perpendicular to it The origin of the co-ordinate system is at the lowest point of the flat plate. An uniform magnetic field of strength H_0 is applied perpendicular to the plate. Thus the equations which describe the unsteady hydromagnetic free convection flow of a viscous incompressible fluid past an infinite flat plate are :

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = k\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \qquad \ldots (1)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = g\beta(\bar{T}-\bar{T}_\infty) - \frac{\sigma_1 \bar{B}_0^2}{\rho}\,\bar{u} + \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \qquad \ldots (2)$$

where $\bar{u}$ is the velocity along the plate, $\bar{v}$ a non-zero variable suction velocity, such as $\bar{v} = \bar{v}\ (1+\epsilon A \exp(i\bar{\omega}\bar{t}))$, v a non-zero negative constant suction velocity, ϵ is a very small parameter ($\epsilon A \leqslant 1$), ω the frequency of oscillations, $\bar{t}$ the time variable, T the temperature in the boundary layer, T_∞ the temperature at a large distance from the plate, $\bar{B}_0\,(= \mu\,\bar{H}_0)$ the magnetic induction, σ_1 the electrical conductivity of the fluid, ρ the density, ν the coefficient of kinematic viscosity, μ the magnetic permeability of the fluid, k the coefficient of thermal conductivity, g the acceleration due to gravity, β the coefficient of volume expansion. This problem has been discussed by Pop (1969) when the suction velocity is constant and in this paper it has been extended for variable suction velocity.

To reduce the basic equations into non-dimensional forms, we introduce as usual in such problems

$$\left.\begin{aligned} u &= \frac{\bar{u}}{|V|}, \quad y = \frac{\bar{y}\,|\bar{v}_s|}{\nu}, \quad t = \frac{\bar{v}_s^2 \bar{t}}{\nu}, \\ T &= \frac{\bar{T}-\bar{T}_\infty}{\bar{T}_w-\bar{T}_\infty}, \quad G = \frac{g\beta\nu(\bar{T}_w-\bar{T}_\infty)}{|\bar{v}_s^3|} \\ \sigma &= \mu/k, \quad M = \frac{\sigma_1 \bar{B}_0^2 \nu}{\rho \bar{v}_s^2}, \quad \omega = \frac{\omega\nu}{\bar{v}_s^2} \end{aligned}\right\} \quad \ldots (3)$$

where $\bar{T}_w$ is the mean wall temperature, G the Grashof number; σ the Prandtl number and M the hydromagnetic parameer.

Using equations to set (3) into (1) and (2), we have

$$\frac{\partial^2 T}{\partial y^2} + \sigma(1+\epsilon A \exp(i\omega t)) \frac{\partial T}{\partial y} = \sigma \frac{\partial T}{\partial t} \quad \ldots (4)$$

and

$$\frac{\partial^2 u}{\partial y^2} + (1+\epsilon A \exp(i\omega t)) \frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} = -GT + Mu \quad \ldots (5)$$

The boundary conditions for u and T are :

$$\left.\begin{aligned} y = 0 : \quad & u = 0, \quad T = 1+\epsilon \exp(i\omega t) \\ y = \infty : \quad & u = 0, \quad T = 0. \end{aligned}\right\} \quad \ldots (6)$$

2. Solution of Equation

In order to solve the equations (4) and (5), with the boundary conditions of set (6), we assume following series for $u(y, t)$ and $T(y, t)$:

$$u(y, t) = F_0(y) + \epsilon F_1(y) \exp(i\omega t), \quad \ldots (7)$$

$$T(y, t) = T_0(y) + \epsilon T_1(y) \exp(i\omega t). \quad \ldots (8)$$

The usual convection is that the real parts in above equations (7) and (8) as well in the suction velocity have physical meaning. Substituting the equations (7) and (8) into (4) and (5) and separating the terms independent of $\epsilon \exp(i\omega t)$ and the coefficients of $\epsilon \exp(i\omega t)$, we have the following set of equations :

$$\frac{d^2 T_0}{dy^2} + \sigma \frac{dT_0}{dy} = 0, \quad \ldots (9)$$

$$\frac{d^2 T_1}{dy^2} + \sigma \frac{dT_1}{dy} - \sigma i\omega T_1 = -\sigma A \frac{dT_0}{dy}, \quad \ldots (10)$$

$$\frac{d^2F_0}{dy^2}+\frac{dF_0}{dy} = -GT_0+MF_0, \qquad \dots (11)$$

$$\frac{d^2F_1}{dy^2}+\frac{dF_1}{dy} -(i\omega+M)F_1 = -GT_1-A\frac{dF_0}{dy}, \qquad \dots (12)$$

with the reduced boundary conditions

$$\left.\begin{array}{lll} y=0: & T_0=T_1=1; & F_0=F_1=0, \\ y=\infty: & T_0=T_1=0; & F_0=F_1=0. \end{array}\right\} \qquad \dots (13)$$

Solving above equations, we have

$$T_0(y) = \exp(-\sigma y), \qquad \dots (14)$$

$$T_1(y) = \exp(-\sigma hy)+\frac{i\sigma A}{\omega}\,(\exp(-\sigma y)-\exp(-\sigma hy), \qquad \dots (15)$$

$$F_0(y) = \frac{G}{\sigma(\sigma-1)-M}\,(\exp(-\lambda y)-\exp(-\sigma y)), \qquad \dots (16)$$

$$F_1(y) = \frac{G\left(\frac{i\sigma A}{\omega}-1\right)}{\sigma^2h^2-\sigma h-(M+i\omega)}[\exp(-\sigma hy)-\exp(-ly)]$$

$$+\frac{G\sigma A}{\sigma^2-\sigma-(M+i\omega)}\left(\frac{1}{\sigma(\sigma-1)-M}+\frac{i}{\omega}\right)(\exp(-ly-\exp(-\sigma y))$$

$$+\frac{A\lambda G}{\lambda^2-\lambda-(M+i\omega)}[\exp(-\lambda y)-\exp(-ly)], \qquad \dots (17)$$

where

$$\left.\begin{array}{l} h = hr+ihi = \frac{1}{2}\left[1+\left(1+\frac{4i\omega}{\sigma}\right)^{\frac{1}{2}}\right], \\ \lambda = \frac{1}{2}[1+(1+4M)^{\frac{1}{2}}], \\ l = \frac{1}{2}[1+(1+4(M+i\omega))^{\frac{1}{2}}]. \end{array}\right\} \qquad \dots (18)$$

It has been assumed that the denominators of various terms for F_0 and F_1 are different from zero.

In above solutions, we have dropped one of the terms of solutions for $y \to \infty$, because

$$\left.\begin{array}{l} \text{Re } 1-\left(1+\frac{4i\omega}{\sigma}\right)^{\frac{1}{2}} < 0 \\ \text{Re } [1-(1+4i\omega+4M)^{\frac{1}{2}}] < 0. \end{array}\right\} \qquad \dots (19)$$

hold for $\omega \neq 0$ and $M \geqslant 0$. Thus for the temperature and velocity distributions inside the magnetic boundary layers, we have

$$T(y,t) = \exp(-\sigma y)+\epsilon \exp(i\omega t)\ [\exp(-\sigma h y)+\frac{i\sigma A}{\omega}(\exp(-\sigma y)-\exp(-\sigma h y))] \quad \ldots (20)$$

$$u(y,t) = \frac{G}{\sigma(\sigma-1)-M}(\exp(-\lambda y)-\exp(-\sigma y))+\epsilon \exp(i\omega t)\left[\frac{G\left(\frac{i\sigma A}{\omega}-1\right)}{\sigma(\sigma-1)h^2-M}\right.$$
$$\times(\exp(-\sigma h y)-\exp(-ly))+\frac{G\sigma A}{\sigma^2-\sigma-(M+i\omega)}\left(\frac{1}{\sigma(\sigma-1)-M}+\frac{i}{\omega}\right)$$
$$\left.\times(\exp(-ly)-\exp(-\sigma y))+\frac{A\lambda G}{\lambda^2-\lambda-(M+i\omega)}(\exp(-\lambda y)-\exp(-ly)\right] \quad \ldots (21)$$

The non-dimensional heat transfer from the wall to the fluid is

$$q = -G\left(\frac{\partial T}{\partial y}\right)_{y=0}$$
$$= G\sigma\left[\ 1+\epsilon \exp(i\omega t)\left\{h+\frac{i\sigma A}{\omega}(1-h)\right\}\right] \quad \ldots (22)$$

The non-dimensional skin-friction at the wall is

$$\tau_w = \left(\frac{\partial u}{\partial y}\right)_{y=0} = G\left[\frac{\sigma-\lambda}{\sigma^2-\sigma-M}+\epsilon \exp(i\omega t)\left\{\frac{\left(\frac{i\sigma A}{\omega}-1\right)}{\sigma^2h^2-\sigma h-(M+i\omega)}\right.\right.$$
$$\times(l-\sigma h)+\frac{\sigma A}{\sigma^2-\sigma-(M+i\omega)}\left(\frac{1}{\sigma(\sigma-1)-M}+\frac{i}{\omega}\right)(\sigma-l)$$
$$\left.\left.+\frac{\lambda A}{\lambda^2-\lambda-(M+i\omega)}(l-\lambda)\right\}\right] \quad \ldots (23)$$

3. Expansion for Large Frequency

In such problems, it is essential for the frequency of oscillations to be large and so for large frequency of oscillations (to avoid the effects of induced magnetic field), we have from equations of set (18) that

$$\left.\begin{aligned} h &\simeq \left(\frac{i\omega}{\sigma}\right)^{\frac{1}{2}} \\ l &\simeq (i\omega)^{\frac{1}{2}} \end{aligned}\right\} \quad \ldots (24)$$

when the magnetic field is fixed.

Thus

$$T_1 \simeq \exp\{-y(i\omega\sigma)^{\frac{1}{2}}\}+\frac{i\sigma A}{\omega}[\exp(-\sigma y)-\exp(-y(i\omega\sigma)^{\frac{1}{2}})] \qquad \ldots (25)$$

$$F_1 \simeq \frac{iG}{(\sigma-1)\omega}[\exp(-y)(i\omega)^{\frac{1}{2}})-\exp(-y(i\omega\sigma)^{\frac{1}{2}})]$$

$$+\frac{G\sigma A}{\sigma^2-\sigma-(M+i\omega)}\left[\frac{1}{\sigma(\sigma-1)-M}\{\exp(-y)(i\omega)^{\frac{1}{2}})-\exp(-y\sigma)\}\right]$$

$$+\frac{A\lambda G}{\lambda^2-\lambda-(M+i\omega)}[\exp(-\lambda y)-\exp(-y(i\omega)^{\frac{1}{2}})] \qquad \ldots (26)$$

Similarly the expressions q and T_w can be reduced for large frequency of fluctuations.

For further discussions, it is desirable to separate, T_1, F_1 into real and imaginary parts. Thus we have

$$T_1 = T_{1r}+iT_{1i} = [\exp(-\sigma yh_r)\cos(y\sigma h_i)-\frac{\sigma A}{\omega}\exp(-\sigma yh_r)\sin(y\sigma h_i)$$

$$+i[\exp(-\sigma yh_r)\sin(\sigma yh_i)-\frac{\sigma A}{\omega}\exp(-\sigma yh_r)\cos(\sigma yh_r)+\frac{\sigma A}{\omega}\exp(-\sigma y)] \qquad \ldots (27)$$

and similarly for F_1 we can break into real and imaginary parts. Now

$$\left.\begin{aligned} T_1 &= (T_{1r}^2+T_{1i}^2)^{\frac{1}{2}}\exp.(i\phi), \\ \text{where} \quad \phi &= \tan^{-1}\frac{T_{1i}}{T_{1r}}. \end{aligned}\right\} \qquad \ldots (28)$$

4. Discussion

If we put $A = 0$, we have the case discussed by Pop (1969) and the corresponding equations for F_0, F_1, T_0 and T_1 are reduced to the equations obtained by Pop.

The variations of T_{1r} and T_{1i} are tabulated and these have been utilized to show the variations of $T_1 = (T_{1r}^2+T_{1i}^2)^{\frac{1}{2}}$ and phase angle ϕ with respect to A in figures 1, 2 and 3. In all figures, we have chosen $\omega = 50$, $\sigma = \frac{1}{2}$. For $y = 0$, we have $T_1 = 1$ and $\phi = 0°$ for all values of A.

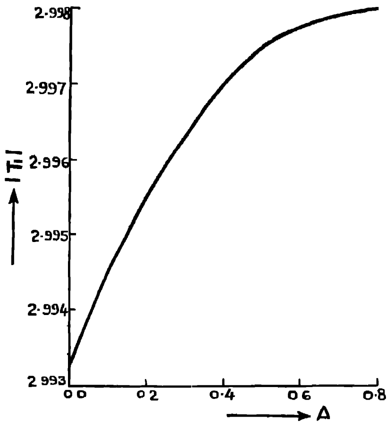

Figure 1. Graph between $|T_1|$ and A when $y = \frac{1}{2}$.

From figure 1, we see that $|T_1|$ and A are increasing together. The increment is uniform but slow when $A \geqslant 0{\cdot}5$.

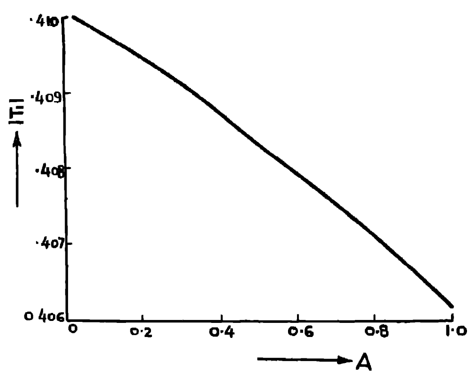

Figure 2. Graph between $|T_1|$ and A when $y = \frac{1}{4}$.

In figure 2, the variations of $|T_1|$ with A have been shown when $y = 1/4$, *i.e.* nearer to the boundary and in this case we see that as A increases the values of $|T_1|$ decreases. Graph between $|T_1|$ and A is almost a straight line and thus there is a linear relation between $|T_1|$ and A.

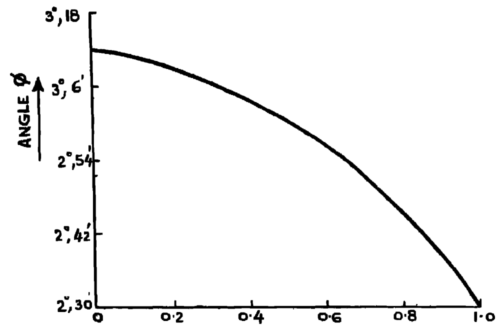

Figure 3. Graph between ϕ and A when $y = \frac{1}{2}$.

In this case we see that the phase angle is positive and decreases as A increases. Thus by increasing the fluctuations of unsteady part of suction velocity, we can decrease the phase angle.

In the same manner, we can draw more graphs for T by selecting certain values of ωt, c, y, σ and A. The same process may be repeated for the velocity components F_1 and u. Further equation (22) has been used and in this case, we have

$$q/G = \sigma+(\sigma\epsilon)\exp(i\omega t)\,|N|\exp(i(\omega t+\xi)), \qquad \ldots (29)$$

where

$$\left.\begin{aligned} |N| &= (N_r^2+N_i^2)^{\frac{1}{2}} \\ \xi &= \tan^{-1}\frac{N_i}{N_r} \end{aligned}\right\} \qquad \ldots (30)$$

Expressions of N_r and N_i after simplifications are

$$\left.\begin{aligned} N_r &= \left(h_r+\frac{\sigma A h_i}{\omega}\right) \\ N_i &= h_i+\frac{\sigma A}{\omega}(1-h_r) \end{aligned}\right\} \qquad \ldots (31)$$

and h_r, h_i are real and imaginary parts of h as used in equation (18).

From the expressions of N_r and N_i, we see that $\xi = 45°$ when $A = 0$. Hence for constant suction velocity, we see that the rate of heat transfer from the wall to the fluid leads by 45° while for variable suction velocity, this phase angle is decreasing as the variable part of the suction velocity increases.

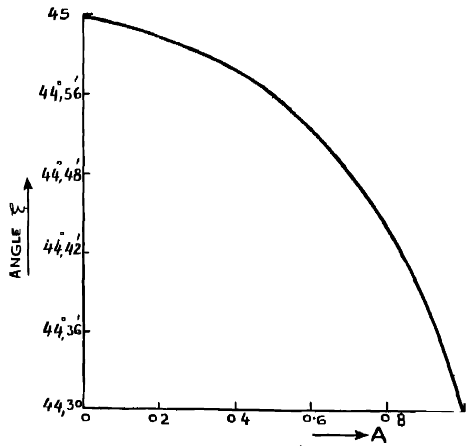

Figure 4. Graph between ξ and A.

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Reference

Pop I 1969 *ZAMM*, 49, Heft 12, 756.